Using differentials to differentiate trigonometric and exponential functions

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Differentiating a polynomial is easy. To differentiate u^2 with respect to u, start by computing

$$d(u^2) = (u + du)^2 - u^2 = 2u\,du + du^2$$

then dropping the last term, an operation that can be justified in terms of limits. Differential notation, in general, can be regarded as a shorthand for a formal limit argument. Still more informally, one can argue that du is small compared to u, so that the last term can be ignored at the level of approximation needed. After dropping du^2 and dividing by du, one obtains the derivative, namely $d(u^2)/du = 2u$. Even if one regards this process as merely a heuristic procedure, it is a good one, as it always gives the correct answer for a polynomial. (Physicists are particularly good at knowing what approximations are appropriate in a given physical context. A physicist might describe du as being much smaller than the scale imposed by the physical situation, but not so small that quantum mechanics matters.)

However, this procedure does not suffice for trigonometric functions. For example, we may write

$$d(\sin\theta) = \sin(\theta + d\theta) - \sin\theta = \sin\theta\Big(\cos(d\theta) - 1\Big) + \cos\theta\sin(d\theta),$$

but to go further we must know something about $\sin\theta$ and $\cos\theta$ for small values of θ. Exponential functions offer a similar challenge, since

$$d(e^\beta) = e^{\beta+d\beta} - e^\beta = e^\beta(e^{d\beta} - 1),$$

and again we need additional information, in this case about e^β for small values of β.

One solution is to use the squeeze lemma to derive the necessary properties of the trigonometric functions, and the limit definition of e for the exponential function. An alternative in the latter case is to define the exponential function as the solution of the appropriate differential equation (see [1]) or as the inverse of the natural logarithm, the latter defined through integration. Modern courses often omit these details, rightly regarded as tedious (and largely irrelevant) by students, replacing them with numerical estimates of the slopes of the corresponding graphs.

We present here an alternative construction starting from geometric definitions, using only infinitesimal reasoning, without limits, numerical estimates, differential equations, or integration.

Circle Trigonometry

The basic (circular) trigonometric functions can be defined geometrically in terms of points (x, y) on the circle of radius r by

$$\cos\theta = \frac{x}{r} \tag{1}$$

$$\sin\theta = \frac{y}{r} \tag{2}$$

where the angle θ is defined as the ratio of the corresponding arc length to the radius. With this definition, the fundamental identity $\cos^2\theta + \sin^2\theta = 1$ follows from the definition of a circle.

We want to differentiate these functions. What do we know? We know that (infinitesimal) arc length along the circle is given by $ds = r\,d\theta$, and we also have the (infinitesimal) Pythagorean theorem, $ds^2 = dx^2 + dy^2$. Furthermore, from $x^2 + y^2 = r^2$, we obtain
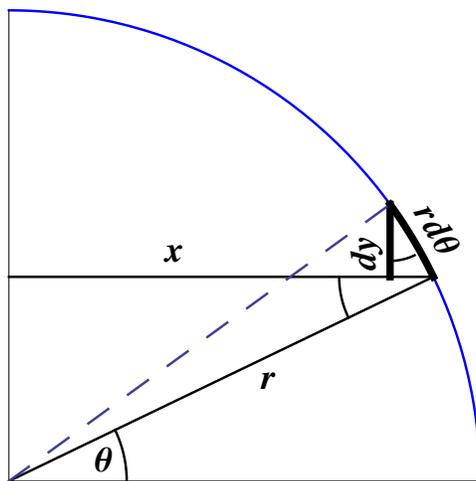
$$x\,dx + y\,dy = 0, \tag{3}$$

Figure 1: $d(r\sin\theta) = dy = \dfrac{x}{r}\, r\, d\theta = r\cos\theta\, d\theta$.

because we can differentiate polynomials. Putting this information together,

$$r^2\, d\theta^2 = dx^2 + dy^2 = dx^2\left(1 + \frac{x^2}{y^2}\right) = r^2\,\frac{dx^2}{y^2},$$

so that, using (3),

$$d\theta^2 = \frac{dx^2}{y^2} = \frac{dy^2}{x^2}.$$

Carefully referring to our circle to check signs, we take the square root and rearrange terms to obtain

$$\begin{aligned} dy &= x\, d\theta \\ dx &= -y\, d\theta. \end{aligned}$$

Finally, inserting (1) and (2) and using the fact that r is constant, we recover the familiar expressions

$$\begin{aligned} d\sin\theta &= \cos\theta\, d\theta \\ d\cos\theta &= -\sin\theta\, d\theta, \end{aligned}$$

and we have differentiated the basic trigonometric functions using little more than their geometric definition, the Pythagorean theorem, and the ability to differentiate simple polynomials. Alternatively, this style of infinitesimal calculation is displayed in the proof-without-words form in Figure 1; a similar construction was given in [6].
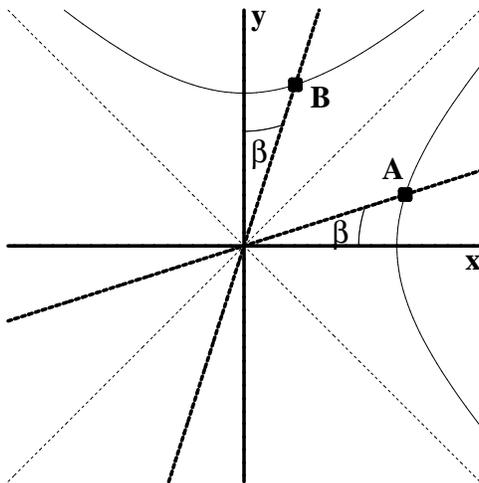
Figure 2: The hyperbolic trigonometric functions defined in terms of hyperbolas of constant radius in Minkowski space.

Hyperbola Trigonometry

The derivation in the previous section carries over virtually unchanged to hyperbolic trigonometric functions. However, as the geometric context is less well known, we repeat the argument with appropriate modifications.

The Lorentzian measure of the squared distance of the point with coordinates (x, y) from the origin is given by

$$\rho^2 = x^2 - y^2,$$

which for the moment we assume to be positive. Curves of constant squared distance from the origin are hyperbolas, and we first consider the branch with $x > 0$. If A is a point on this hyperbola, then we can define the hyperbolic angle β between the line from the origin to A and the (positive) x-axis to be the ratio of the *Lorentzian* length of the arc of the hyperbola between A and the point $(\rho, 0)$ to the "radius" ρ, where Lorentzian length is obtained by integrating $d\sigma$, where $d\sigma^2 = |dx^2 - dy^2|$. (See Figure 2.)

This Lorentzian geometry is known as *Minkowski space*, and is the geometry of special relativity [2, 3]. Because hyperbolas in Minkowski space play the role that circles do in Euclidean geometry, one could also call this geometry *hyperbola geometry* [3].

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We define the hyperbolic trigonometric functions in terms of the coordinates (x, y) of A, that is

$$\cosh \beta = \frac{x}{\rho} \qquad (4)$$

$$\sinh \beta = \frac{y}{\rho} \qquad (5)$$

This construction is shown in Figure 2, which also contains another hyperbola, $x^2 - y^2 = -\rho^2$. By symmetry, the point B on the latter hyperbola has coordinates $(\rho \sinh \beta, \rho \cosh \beta)$.

Many of the features of these functions follow immediately from these geometric definitions. Since the minimum value of x/ρ on the hyperbola is 1, we have $\cosh \beta \geq 1$. As β approaches $\pm\infty$, x approaches ∞ and y approaches $\pm\infty$, agreeing with the asymptotic behavior of the graphs of $\cosh \beta$ and $\sinh \beta$, respectively.

We differentiate these functions using the same technique as before. We choose to work on the hyperbola $y^2 - x^2 = \rho^2$, the one in Figure 2 that contains B. What do we know? We know that (infinitesimal) arc length along the hyperbola is given by

$$ds = \rho \, d\beta,$$

but we also have the (infinitesimal, Lorentzian) Pythagorean theorem

$$ds^2 = dx^2 - dy^2.$$

Furthermore, from $y^2 - x^2 = \rho^2$ we obtain

$$x \, dx = y \, dy. \qquad (6)$$

Putting this information together,

$$\rho^2 \, d\beta^2 = dx^2 - dy^2 = dx^2 \left(1 - \frac{x^2}{y^2} \right) = \rho^2 \frac{dx^2}{y^2},$$

so that
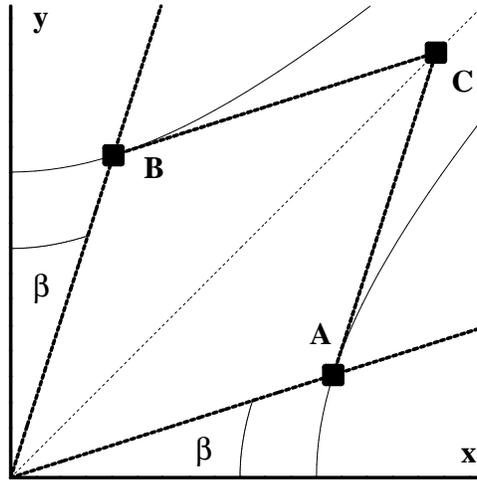
$$d\beta^2 = \frac{dx^2}{y^2} = \frac{dy^2}{x^2},$$

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Figure 3: The geometric definition of $\exp(\beta)$. Points A and B are as in Figure 2, so that the coordinates of point C are $(\rho \exp(\beta), \rho \exp(\beta))$.

where the last step uses (6). Using Figure 2 to check signs, we take the square root and rearrange terms to obtain

$$\begin{aligned} dy &= x \, d\beta \\ dx &= y \, d\beta. \end{aligned}$$

Finally, inserting (4) and (5) and using the fact that $\rho = $ constant, we obtain

$$\begin{aligned} d \sinh \beta &= \cosh \beta \, d\beta \\ d \cosh \beta &= \sinh \beta \, d\beta, \end{aligned}$$

thus differentiating the basic hyperbolic trigonometric functions.

Exponentials (and Logarithms)

The hyperbolic functions are usually defined by

$$\cosh \beta = \frac{e^\beta + e^{-\beta}}{2}$$

and
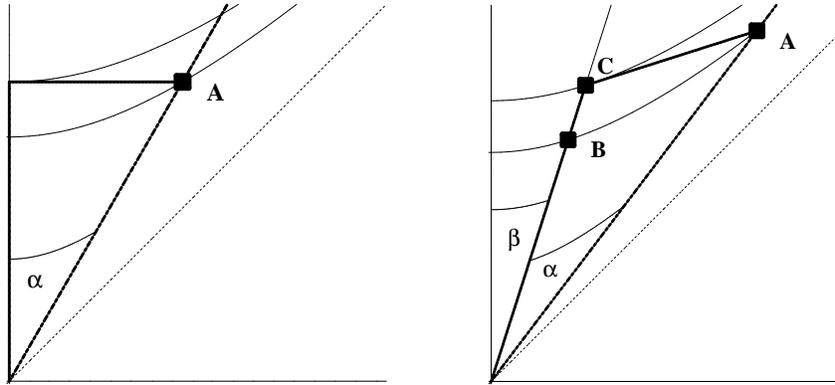
$$\sinh \beta = \frac{e^\beta - e^{-\beta}}{2},$$

Figure 4: The geometric construction of the hyperbolic addition formulas.

and it takes some work (and independent knowledge of the exponential function) to show directly that our definition is equivalent to this one. We turn this on its head and instead define $\exp(\beta)$ by

$$\exp(\beta) = \cosh\beta + \sinh\beta. \tag{7}$$

Our $\exp(\beta)$ has immediately the geometric interpretation shown in Figure 3, where it is important to recall that β is a *hyperbolic* angle, not a Euclidean angle as measured by a (Euclidean) protractor. We also immediately have

$$d\big(\exp(\beta)\big) = \sinh\beta\,d\beta + \cosh\beta\,d\beta = \exp(\beta)\,d\beta. \tag{8}$$

Since clearly

$$\exp(0) = 1$$

we can conclude that our $\exp(\beta)$ is the same as e^{β} by invoking uniqueness results for solutions of differential equations.

It is also possible to show directly that $\exp(\beta)$ is an exponential function, that is, it satisfies

$$\exp(\alpha + \beta) = \exp(\alpha)\,\exp(\beta). \tag{9}$$

One way uses the well-known fact that the derivative of a vector $\boldsymbol{v}$ of constant magnitude is perpendicular to that vector. Since the vector to points on a circle from its center has constant magnitude, it follows that radii are orthogonal to circles. The argument holds just as well in Minkowski space, and establishes (Lorentzian) orthogonality between radial lines to our hyperbolas and the tangents to these hyperbolas. Thus, the two right triangles
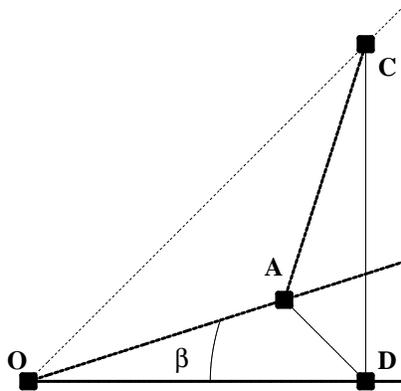
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Figure 5: The geometric verification that exp is exponential.

shown by the heavy lines in Figure 4 are congruent in Minkowski space, from which it is straightforward to derive the standard formulas for $\sinh(\alpha + \beta)$ and $\cosh(\alpha + \beta)$. These also establish (9). (For further details, see [3].) A direct geometric verification is also possible, based on the construction in Figure 3, shown in more detail in Figure 5. Denoting the origin by O, and noting that line AD is at $45°$, we see that the ratio of the length of OD to that of OA is precisely $\exp(\beta)$. In this sense, rotation through β (taking A to D) corresponds to stretching by a factor of $\exp(\beta)$. Composing two such rotations leads directly to (9); the details are left to the reader.

Having verified the desired properties of the exponential function, it is straightforward to define logarithms as the inverse of exponentiation, that is, to define $\log(u) = \beta$ if and only if $u = \exp(\beta)$, and establish that this definition leads to the usual properties of the natural logarithm.

Conclusion

It is remarkable that the elementary argument we have presented leads to correct formulas for the derivatives of the trigonometric functions. Furthermore, it is advantageous that our argument models a fundamental aspect of differential calculus, the art of infinitesimal reasoning. We believe students will find this argument more enlightening than standard treatments. The differentiation step needed is within their reach; the use of differentials turns it into a simple, single-variable calculus computation without partial derivatives or implicit differentiation.

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Unfortunately, the same can not really be said about our treatment of the exponential function, since Lorentzian geometry, although interesting, has little place in a calculus course. Our definition of $\exp(\beta)$, however elegant, is therefore unlikely to be much use even to calculus instructors wishing to use differentials. Nonetheless, we have provided a path to the derivatives of both trigonometric and exponential functions without the use of limits, numerical estimates, solutions of differential equations, or integration.

The key geometric idea underlying all of the results in this paper is the fact that the derivative of a vector of constant magnitude is orthogonal to the original vector. The two basic trigonometric functions (either circular and hyperbolic) are the components of vectors of constant magnitude. In two dimensions, orthogonal vectors are obtained by swapping components and inserting an appropriate minus sign. Thus, the derivatives of each basic trigonometric function must be the other, with an appropriate minus sign added according to whether one is in Euclidean geometry or Minkowski space.

Acknowledgments

Summary

Starting from geometric definitions, we show how differentials can be used to differentiate trigonometric and exponential functions without limits, numerical estimates, solutions of differential equations, or integration.

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